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Mobility of Electrons in the Noble Gases*

JOSEPH C. BOWE
Radiological Physics Division, Argonne National Laboratory, Argonne, Illinois
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This paper presents a theoretical study which leads to formulas that relate the cross section for inelastic collisions near the threshold energy to the drift velocity v_d of electrons in noble gases. Derivation of general formulas for v_d follows a qualitative interpretation of the main features of the experimental drift-velocity curves. The formulas for v_d are derived for the case in which the elastic cross section for momentum transfer is given by $N_0Q_m(v)=av^{j-1}$. Closed-form expressions for v_d are obtained by integrating only over the distribution function for electrons that have energy less than the excitation energy u_1. By this procedure, integration over the high-energy $(u>u_1)$ distribution function is avoided and the use of analytical methods is made possible. A partial correction to this approximation is obtained by extending the low-energy $(u<u_1)$ distribution function to pass through zero at an energy u_0 that is greater than u_1. From the drift-velocity formula and the experimental values of v_d, the cutoff energy u_0 can, in principle, be evaluated as a function of E/p. The high-energy distribution function is used to derive an expression for the overshoot (u_0-u_1) which is shown to be proportional to $(E/p)^{2/(\gamma+2)}$ and to depend upon the constants h and γ when the inelastic-collision cross section has the form $Q_1(u)=h(u-u_1)^\gamma$. Possible applications of these theoretical results to drift-velocity data for gas mixtures are also briefly discussed.

I. INTRODUCTION

THE drift velocity of electrons in the noble gases has a remarkably simple dependence (Fig. 1) upon E/p (the ratio of applied field to gas pressure) that bears a direct relation to the fundamental collision processes between electrons and gas atoms. It is therefore of interest to show how these characteristics of the drift-velocity data peculiar to the noble gases can be understood from theory. In particular, it is desired to obtain a better understanding of the effect that inelastic collisions have upon the drift velocity. The striking up-turn in the drift-velocity curve is assumed to be closely correlated to the cross section for inelastic collision. By making a more thorough study of the role of inelastic collisions than was made in earlier publications,[1] this paper derives formulas in closed form by which the cross section for excitation near threshold can, in principle, be deduced from drift-velocity data. The present paper adds to and extends the previously published work by obtaining formulas for the drift velocity in terms of a cutoff energy u_0 and by finding a relation between u_0 and the cross section for inelastic collisions.

The drift velocity of a swarm is a macroscopic quantity, whereas the collision cross section is an atomic quantity. The macroscopic properties, however, are determined by the atomic properties and the interdependence of the two is formulated by statistical theory in which the Maxwell-Boltzmann transport equation defines the velocity distribution for electrons. Analytical formulas that relate the drift velocity to the atomic cross section for elastic collisions have indeed already been derived for the region of low E/p where inelastic collisions are neglected.[2] For the region of

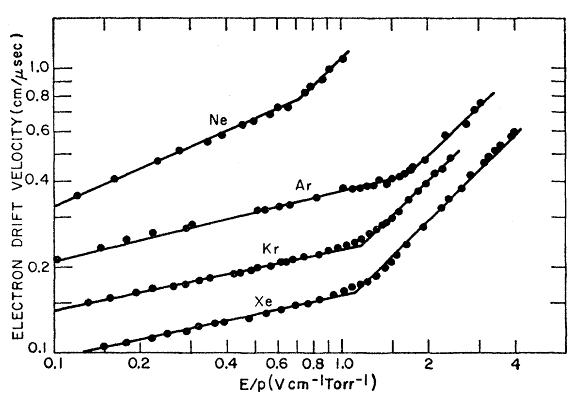

FIG. 1. Experimental drift-velocity curves for electrons. Inelastic collisions cause the slope to abruptly assume the value unity at $(E/p)^*$. Data for helium are not shown because the break in this curve is partly due to the decreasing cross section for elastic collisions above a few eV. (Notice that the plot is on a log-log scale.)

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[1] P. Walsh, Bull. Am. Phys. Soc. **7**, 632 (1962); D. Barbiere, Phys. Rev. **84**, 653 (1951); H. W. Allen, *ibid.* **52**, 707 (1937); M. J. Druyvesteyn, Physica **4**, 464 (1937).

[2] J. C. Bowe, Phys. Rev. **117**, 1416 (1960).

higher E/p, similar expressions that relate the drift velocity directly to the cross section for inelastic collisions have not heretofore been obtained because of the mathematical difficulties that arise.

Frost and Phelps,[3] assuming a set of cross sections for the rotational and vibrational excitation of the *molecular* gases H_2 and N_2, used a digital computer to evaluate the mobility and diffusion coefficients from the Boltzmann transport theory. They adjusted the input cross sections to obtain a good fit between the computed and experimental values of mobility and the average energy of the electron swarm. In this way, they obtained information about the cross sections throughout the energy range from about 0.006 to 2 eV.

The present paper follows analytical procedures used by Allis and Brown.[4] In addition to providing a clearer insight into the theory, the analytical approach leads to new expressions in closed form by which the cutoff energy u_0 can be evaluated from experimental drift-velocity data and by which the inelastic cross section can be evaluated from u_0. This study also suggests a possible quantitative explanation of the effect that small admixtures of a molecular gas have on the drift velocity of electrons in the noble gases.

The technique of applying statistical theory to swarm measurements for the purpose of evaluating atomic cross sections does not necessarily yield values that are unique. But the values that are obtained often represent the best information available in energy ranges that are not currently accessible by the more direct methods of measurement. In those energy ranges where the cross sections are accurately known by direct measurement, the values obtained from swarm measurements serve to test the statistical theory.

The formulas derived in this paper are obtained in closed form by using the distribution function with which Allis and Brown were so successful: That is, the low-energy ($u<u_1$) function obtained from the Boltzmann equation is extended to energies greater than the excitation energy u_1. In this approximation, the extended function passes through zero at an unspecified energy u_0 and the high-energy ($u>u_1$) function is not used for computations of mean values. Section III derives formulas that can be used to evaluate u_0 from the drift-velocity data. In Sec. IV, the method of Allis[5] is followed to derive an expression for the overshoot (u_0-u_1) when the cross section for inelastic collisions is given by $Q_1(u)=h(u-u_1)^\gamma$. The overshoot is found to be proportional to $(E/p)^{2/(\gamma+2)}$ and to depend upon the constants h and γ. Section II presents the salient features of the experimental drift-velocity data, a qualitative interpretation of them, and an outline for a quantitative interpretation. The use of drift-velocity data to obtain information about collision cross sections is discussed, and it is pointed out how the derived formulas might also be applied to data in gas mixtures.

[3] L. S. Frost and A. V. Phelps, Phys. Rev. **127**, 1621 (1962).

[4] W. P. Allis and S. C. Brown, Phys. Rev. **87**, 419 (1952).

[5] W. P. Allis, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21, p. 417.

II. DRIFT-VELOCITY DATA

Experimental drift-velocity curves for electrons in the noble gases can be divided into three distinct regions of E/p. At very low values of E/p (usually below about 0.1 V cm^{-1} Torr^{-1}) the drift velocity v_d increases linearly with E/p.[6] The curves shown in Fig. 1 for larger values of E/p have a sharp break at a well-defined value $(E/p)^*$ that is different for each gas.[7] The portion of each curve below $(E/p)^*$ has a slope that is less than unity. The functional relationship between v_d and E/p in this region of E/p is correlated by theory to the cross section $Q_m(u)$ for momentum transfer in an elastic collision between an electron of energy u and a gas atom. Thus, if $Q_m(u)$ is constant and independent of energy, theory predicts that v_d should increase as the one-half power of E/p. This relation is experimentally realized in helium and is found to be very nearly true for neon over limited ranges of E/p. If $Q_m(u)$ increases linearly with energy, theory indicates that v_d is proportional to the one-fourth power of E/p. This relation is found experimentally in argon, krypton, and xenon over restricted ranges of E/p. Above $(E/p)^*$, the variation of v_d is again linear and therefore the mobility at unit pressure (v_d/E) is again independent of E/p. The present paper develops the theory that relates this behavior to the cross section for excitation. A previous paper[2] related the cross section for elastic collisions to the measurements of v_d below $(E/p)^*$.

These features of the drift-velocity curve can be qualitatively understood by the following argument.[8] At very low values of E/p, the electrons are in thermal equilibrium with the gas atoms and therefore have a Maxwellian energy distribution that is characteristic of the gas temperature T. Under this condition, the applied electric field does not determine the average energy of the electrons and therefore does not affect their average collision rate. It is a consequence even of the elementary theory that when the collision rate is independent of the applied field, v_d increases linearly with E/p. The average power input per electron ($P=eEv_d=\mu eE^2$, where μ is the mobility) is dissipated by the energy losses due to elastic collisions.

At intermediate values of E/p, the rate at which electrons (whose velocity distribution is characteristic of the ambient gas temperature) lose energy by elastic collisions is not sufficiently large to balance the rate of energy input from the electric field. Therefore, to restore the power balance at a given E/p, the random energy of the electrons must increase. It is an experimental fact

[6] J. L. Pack and A. V. Phelps, Phys. Rev. **121**, 798 (1961); J. L. Pack, R. E. Voshall, and A. V. Phelps, *ibid.* **127**, 2084 (1962).

[7] J. C. Bowe, Phys. Rev. **117**, 1411 (1960).

[8] This discussion is similar to the one given by W. Shockley, Bell System Tech. J. **30**, 990 (1951), Sec. 3e, in connection with Ohm's law in germanium.

that the energy does indeed increase to many times the value of kT in the gas. A second observed effect is that the mobility is no longer constant, but becomes dependent upon the electric field. If the collision cross section *does not decrease* with energy, the mobility decreases with E/p and thereby helps to restore the power balance. Under this condition, the electrons need not attain as large an average energy as they otherwise would if the cross section decreased with energy. Also, in this range of E/p, the energy distribution is no longer Maxwellian but assumes the form of a *generalized* Druyvesteyn distribution.[9] A singular case occurs, however, when the collision cross section *decreases* inversely as the velocity because then the collision rate is again independent of energy and therefore likewise independent of E/p. In this case, the mobility remains independent of E, v_d increases linearly with E/p, and although the average electron energy increases beyond the value of kT in the gas, the distribution is still Maxwellian but corresponds to a higher effective temperature.

Above the critical value $(E/p)^*$, collisions that excite the first level of the atoms are sufficiently numerous to be effective in maintaining the power balance. Inelastic collisions tend to stabilize the random energy of the electrons and therefore a linear relation between v_d and E/p is again observed, at least in the vicinity immediately above $(E/p)^*$.

Knowledge of the energy distribution provides the link to a quantitative interpretation of these phenomena in terms of the collision cross sections. If the distribution function is known, average values of any quantity pertaining to the electron swarm can, in principle, be computed. Inelastic collisions modify the distribution function not only in the high-energy region, but also in the energy range in which collisions are purely elastic. If, for example, the inelastic collision cross section for excitation of the first level is momentarily imagined to be infinitely large, then clearly no electron would ever attain an energy greater than the first excitation energy u_1 and the distribution function would on that account be cut off at this energy. Electrons that reach energy u_1 would immediately be reduced to zero energy. The average energy of the swarm would not increase rapidly with a further increase in E/p above $(E/p)^*$ if the number of inelastic collisions is not too large. Therefore, qualitatively speaking, v_d should increase linearly with E/p, in the region immediately above $(E/p)^*$. Eventually, at larger values of E/p for which the number of inelastic collisions becomes significantly larger, this linear relation might not prevail.

The cross section for excitation, however, is not infinitely large and hence, the distribution function is not expected to be cut off at u_1. Therefore, it is generally necessary to find the distribution functions separately for the high-energy $(u>u_1)$ and low-energy $(u<u_1)$ regions.[10] In the approximation used in this paper, a formula for drift velocity is obtained in terms of the cutoff energy u_0 and the electron collision frequency $\nu(v_0)$ evaluated at cutoff. The high-energy function, on the other hand, is used to relate the overshoot to the cross section for inelastic collisions by imposing the condition that the two distribution functions must join smoothly at u_1.

The drift-velocity formulas that are obtained in Sec. III are expected to be valid also for gas mixtures if the concentration of the added gas is low enough so that the collision frequency is primarily determined by the bulk gas. Thus, if the added gas has a low-lying excitation level, it could be effective in cutting off the distribution function near that energy. This cutoff mechanism qualitatively explains the "resonance" that is observed in the drift-velocity data for argon that contains minute quantities of a molecular gas. The sharp peak[7] observed at $E/p<0.01$ V cm^{-1} Torr^{-1} in the drift-velocity curve for contaminated argon has been associated with the Ramsauer minimum that the collision cross section has in argon; but the peak is not observed in the pure gas. Thus, it seems that the molecular gas tends to sharpen the energy distribution of electrons by providing a low-energy cutoff. As E/p is increased, the cutoff energy increases and eventually moves across the energy that corresponds to the Ramsauer minimum. As this happens, the collision frequency of the fastest electrons passes through a minimum value and hence the drift velocity passes through a maximum and thereby reflects the variation of the cross section with energy.

The formulas derived in this paper also suggest a method by which information might be obtained about collision processes (e.g., electron attachment) that occur in a second gas that is added in small quantities to the noble gas. Although inelastic collisions with the second gas are not numerous, they should not be ignored because they can be effective in cutting off the energy distribution of electrons at a finite value u_0 when all collisions with the bulk gas are elastic. An electron that loses a relatively large energy in an inelastic collision will not readily regain it from the electric field. Therefore, the drift velocity and average energy of the swarm should in this case be computed by evaluating the integrals over the energy range from zero to u_0 rather than to infinity. Only the cross section for momentum transfer of the bulk gas is needed in these integrals if the total rate of collisions with the second gas is small. The method of obtaining u_0 from the drift-velocity data is detailed in Sec. III. Thus, the bulk gas in effect provides a swarm of electrons of known energy distribution with which to study collision processes in the added gas.[11]

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[9] M. J. Druyvesteyn, Physica **10**, 61 (1930). The distribution function that is obtained when the cross section for elastic collisions is energy-dependent is frequently referred to as the *generalized* Druyvesteyn function.

[10] T. Holstein, Phys. Rev. **70**, 367 (1946).

[11] R. H. Ritchie and G. W. Whitesides, Oak Ridge National Laboratory Report-3081, 1961 (unpublished), expressed the same idea but neglected the effects of inelastic collisions with the second gas.

III. DRIFT-VELOCITY THEORY

A. General

The velocity distribution function $f(\mathbf{v},t)$ for electrons that are in collision equilibrium with a gas is defined by the Boltzmann transport equation

$$(eE/m)(\partial f/\partial v_x)=(\partial f/\partial t)_{\text{collisions}}, \quad (1)$$

where (e/m) is the charge-to-mass ratio of the electron and E is a uniform dc electric field that is applied in the negative x direction. The diffusion term, which is omitted from this equation, need not be included when the gas pressure is sufficiently high. By the method of Lorentz,[12] in which collisions between electrons are ignored, the solution is given in the form

$$f(\mathbf{v})=f_0(v)+(v_x/v)f_1(v), \quad (2)$$

where $f_1(v)$ is presumed to be much smaller than $f_0(v)$. The two functions are related by the equation

$$(eE/m)(df_0/dv)=-vNQ(v)f_1(v), \quad (3)$$

where $Q(v)=Q_{1m}(v)+Q_m(v)$ is the sum of cross sections for momentum transfer in an inelastic[13] and elastic collision, respectively, and $N=N_0p$ is the particle density (scattering centers per cm^3) of the gas at pressure p and 0°C.

The differential equation, which is obtained from Eq. (1) after the right-hand member is expressed explicitly in terms of the cross sections for collisions and which defines $f_0(v)$ for all values of speed,[10] is

$$\tfrac{1}{3}(eE/m)^2(d/dv)\{[v/NQ(v)]df_0/dv\} + (m/M)(d/dv)[v^4NQ_m(v)f_0(v)] = Nv[v^2Q_1(v)f_0(v)-v'^2Q_1(v')f_0(v')], \quad (4)$$

where $Q_1(v)$ is the ordinary total cross section for inelastic collisions. The inelastic cross section is zero for electrons that have speed v less than $v_1=(2u_1/m)^{1/2}$, and therefore, in this region, the solution to Eq. (4) is[4]

$$f_0(v)=Kf_D(v)\int_v^{v_0}[NQ_m(v)/vf_D(v)]dv, \quad (5)$$

where $f_D(v)$ is the solution to the homogeneous equation. Equation (5) assumes that all inelastic collisions occur very close to energy u_1. The effect of these collisions is included in the constant K. The second constant of integration is the speed v_0 at which the distribution function is set equal to zero. This speed is expected to be greater than v_1 because the cross section for inelastic collisions is not infinitely large.

The drift velocity v_d is[14]

$$v_d=-(4\pi eE/3mn)\int_0^\infty[1/NQ(v)][\partial f_0/\partial v]v^2dv, \quad (6)$$

which, after integration by parts, becomes

$$v_d=-\frac{4\pi eE}{3mn}\left\{\left[\frac{v^2}{NQ(v)}f_0(v)\right]_0^\infty - \int_0^\infty \frac{d}{dv}\left[\frac{v^2}{NQ(v)}\right]f_0(v)dv\right\}, \quad (7)$$

where n is the normalization integral $\int f_0(v)\,d\kappa$ taken over all velocity space.

From this equation, the following generalizations can be made independently of the particular form of $f_0(v)$. (1) If $Q(v)=\mathrm{a}v^2$, the integral in Eq. (7) vanishes and the drift velocity is obtained by evaluating the normalized distribution function at $v=0$. (2) In the singular case in which the *total* collision frequency $\nu=N_0vQ(v)$ at unit pressure is independent of speed, the drift velocity is independent of the mathematical form of $f_0(v)$ and varies as the first power of E/p, i.e., $v_d=(e/m)(1/\nu)\times(E/p)$. (3) The factors that multiply (eE/m) in Eq. (7) have the dimension of time. Thus, the experimental values of drift velocity can be interpreted as defining an average "effective" time between collisions as a function of E/p. The reciprocal of this quantity is the "effective" collision frequency ν_m for momentum transfer[3] due to both elastic and inelastic collisions.

For very low values of E/p, the electrons are in thermal equilibrium with the gas and have a Maxwell-Boltzmann velocity distribution that is independent of E/p. In this case, the "effective" collision time has the constant value characteristic of the gas temperature; and Eq. (7) shows that under this condition v_d is proportional to the first power of E/p. When this linear relation is observed experimentally, investigators then know that E/p is sufficiently low for the electrons to be in thermal equilibrium with the gas.

The drift velocity can also be computed directly from power-balance consideration; i.e., the rate at which the electric field supplies energy to the electrons can be equated to the rate at which these electrons lose energy in collisions. The average power input per electron is[3,13]

$$eEv_d=(4\pi/n)\int_0^\infty(2m/M)(mv^2/2)[vNQ_m(v)]f_0(v)v^2dv + (4\pi u_1/n)\int_0^\infty[vNQ_1(v)]f_0(v)v^2dv, \quad (8)$$

[12] H. A. Lorentz, *The Theory of Electrons* (Dover Publications, Inc., New York, 1952), 2nd ed., p. 267.

[13] S. Altshuler, J. Geophys. Res. **68**, 4707 (1963). Altshuler shows that the inelastic cross section contained in Eq. (3) is the cross section $Q_{1m}(v)$ for momentum transfer associated with an inelastic collision and that it is given by $Q_{1m}(v)=\int q_1(v,\theta)\times[1-(1-v_1^2/v^2)^{1/2}\cos\theta]d\omega'$, where $v_1^2=2u_1/m$. Notice, however, that $Q_{1m}(v)$ is equal to the ordinary total cross section $Q_1(v)$ for inelastic collisions for those cases in which the electrons have energy u_1 before collision or when the inelastic scattering is isotropic. In the present paper, the $Q_1(u)$ that appear in the equations of Sec. IV are precisely the ordinary total cross sections for inelastic collisions and contain no approximations that involve either $Q_{1m}(u)$ or the angular dependence of $q_1(u,\theta)$. But in Eq. (4), the term containing $Q_1(v')$ was obtained by using the assumption that inelastic scattering is isotropic.

[14] Reference 5, p. 413.

where the two terms on the right are the average rates of energy loss due to elastic and inelastic collisions, respectively. The inelastic power losses are usually not computed because $f_0(u)$ and $Q_1(u)$ are not accurately known for energies greater than u_1. These losses can, however, be obtained as a function of E/p from Eq. (8) if the elastic-collision power-loss integral is evaluated by computation and if the left-hand member, i.e., the power input per electron, is calculated from the experimental drift-velocity data.

When inelastic collisions occur, the power-balance equation emphasizes the high-energy portion of the distribution function, which is not accurately known, and therefore it is better to compute v_d directly from Eq. (7). Since the main contribution to the drift-velocity integral comes from the electrons that have energy u less than u_1, that portion of the integral in Eq. (7) that extends from u_1 to infinity is omitted and the integration over f_0 is instead extended beyond v_1 to the cutoff value v_0. Thus, the drift velocity is obtained in terms of the cutoff speed v_0 or the cutoff energy u_0.

B. Drift-Velocity Formulas

Drift-velocity formulas that do not ignore inelastic collisions are obtained by integrating the electron velocity over the distribution function given by Eq. (5). These formulas are expected to be reasonably accurate for values of E/p for which inelastic collisions are not too numerous. They are therefore applicable to the drift-velocity data immediately above $(E/p)^*$.

Let the collision probability for momentum transfer at unit pressure be $N_0Q_m(v)=\mathrm{a}v^{j-1}$. The values $j=1$ for neon and $j=3$ for argon, krypton, and xenon give good representations of the cross sections for these gases for energies up to the neighborhood of the excitation energy.[2] For helium, $j=1$ gives a good representation only below a few eV. With this substitution, Eq. (5) becomes

$$f_0(v)=Lpaf_D\int_v^{v_0}(v^{j-2}/f_D)dv\,, \tag{9}$$

where L is a constant and

$$\begin{aligned} f_D&=Ge^{-w}\,,\\ w&=[6m/(j+1)M]\\ &\quad\times[N_0Q_m(v)(eE/p)^{-1}(1/2)mv^2]^2=\beta v^{2j+2}\,,\\ \beta&=[3m/2(j+1)M][map/eE]^2. \end{aligned} \tag{10}$$

Notice that the dimensionless variable w is jointly proportional to (M/m) and the square of the ratio of the average loss that electrons of energy u suffer in an elastic collision to the energy gained from the electric field in a mean free path. In terms of the variable w,

$$f_0(w)=[Lap/(2j+2)]\beta^{(1-j)/(2j+2)}\times e^{-w}\int_w^{w_0}e^{w}w^{-(j+3)/(2j+2)}dw \tag{11}$$

and Eq. (7) becomes

$$\begin{aligned} v_d(E/p,w_0)=&-\frac{4\pi eE}{3mnap}\left[\left(\frac{w}{\beta}\right)^{(3-j)/(2j+2)}f_0(w)\right]_0^{w_0}\\ &+\left(\frac{3-j}{2j+2}\right)\left(\frac{4\pi eE}{3mnap}\right)\beta^{(j-3)/(2j+2)}\\ &\times\int_0^{w_0}w^{(-3j+1)/(2j+2)}f_0(w)dw\,, \end{aligned} \tag{12}$$

where the normalization integral is

$$n=[2\pi/(j+1)]\beta^{-3/(2j+2)}\int_0^{w_0}w^{(1-2j)/(2j+2)}f_0(w)dw\,. \tag{13}$$

Hence, when $j=3$, i.e., when $N_0Q_m=\mathrm{a}v^2$, Eq. (12) becomes

$$v_d(E/p,w_0)=\frac{8eE\beta^{3/8}\displaystyle\int_0^{w_0}w^{-3/4}e^{w}dw}{3map\displaystyle\int_0^{w_0}w^{-5/8}e^{-w}\int_w^{w_0}t^{-3/4}e^{t}dtdw}\,, \tag{14}$$

and $w_0=\frac{3}{2}(m/M)[u_0NQ_m(u_0)/eE]^2$. In the limit of $w_0\ll1$, the integrals are easily evaluated and the result is

$$v_d(E/p,v_0)=\tfrac{5}{2}(e/m)[1/\nu_m(v_0)](E/p)\,. \tag{15}$$

When $j=1$, i.e., when $N_0Q_m=$ constant, Eq. (12) becomes

$$v_d(E/p,w_0)=\frac{2eE\beta^{1/4}\displaystyle\int_0^{w_0}w^{-1/2}e^{-w}\int_w^{w_0}t^{-1}e^{t}dtdw}{3mpN_0Q_m\displaystyle\int_0^{w_0}w^{-1/4}e^{-w}\int_w^{w_0}t^{-1}e^{t}dtdw}\,, \tag{16}$$

and $w_0=(3m/M)(u_0NQ_m/eE)^2$. Here, again in the limit $w_0\ll1$, the drift velocity is

$$v_d(E/p,v_0)=\tfrac{3}{2}(e/m)[1/\nu_m(v_0)](E/p)\,. \tag{17}$$

For physical reasons, it does not seem reasonable that the cutoff energy u_0 should be rigidly clamped as a result of inelastic collisions. On the contrary, it would appear that u_0 should increase when E/p is increased. If this is true, Eqs. (15) and (16) then show that v_d cannot increase linearly with E/p at larger values of E/p for which w_0 is small.

But for the noble gases, the magnitude of $w[(E/p)^*,u_1]$ is greater than unity. In this case, the integrals in Eqs. (14) and (16) are more accurately evaluated in the region of $(E/p)^*$ by an infinite series expansion. The result is

$$v_d(E/p,w_0)=(1/6)(e/m)\times[\nu_m(v_0)]^{-1}[\mathcal{S}_j(w_0)/\mathcal{S}_{j+1}(w_0)](E/p)\,, \tag{18}$$

where for $j=1$, the series are given by

$$\mathcal{S}_1(w_0)=1-\sum_0^\infty \frac{(-1)^k w_0^{k+1}}{(k+1)!(2k+3)^2} +\sum_{k=0}^\infty \sum_{i=0}^k \frac{(-1)^i w_0^{k+1}}{(k-i+1)!i!(2i+1)(2k+3)}, \quad (19)$$

$$\mathcal{S}_2(w_0)=\frac{1}{9}-\sum_0^\infty \frac{(-1)^k w_0^{k+1}}{(k+1)!(4k+7)^2} +\sum_{k=0}^\infty \sum_{i=0}^k \frac{(-1)^i w_0^{k+1}}{(k-i+1)!i!(4i+3)(4k+7)}, \quad (20)$$

and for $j=3$ by

$$\mathcal{S}_3(w_0)=\sum_0^\infty \frac{w_0^k}{k!(4k+1)}, \quad (21)$$

$$\mathcal{S}_4(w_0)=\sum_{k=0}^\infty \sum_{i=0}^k \frac{(-1)^i w_0^k}{(k-i)!i!(8i+3)(8k+5)}. \quad (22)$$

Earlier computations of drift velocity above $(E/p)^*$ lead Allen[1] to conclude that the experimental curves for neon and argon, but not helium, could be fitted with a single constant value of u_0 that is different for the two gases. Allen used the distribution given by Eq. (5) but assumed an elastic cross section that is independent of energy. Equation (18), however, suggests how the cutoff energy u_0 *could* increase with E/p (as it is intuitively expected to do) because the factor $[\nu_m(v_0)]^{-1}[\mathcal{S}_j(w_0)/\mathcal{S}_{j+1}(w_0)]$ might possibly vary with E/p in such a way as to produce a v_d that fits the experimental data. The results of Sec. IV show how the overshoot must indeed increase with E/p. The implications of Allen's result for helium are not clearly understood and therefore further computations should be carried out for this gas.

Equation (18) and the experimental drift-velocity data determine the cutoff energy u_0 as a function of E/p in the following way. The collision frequency $\nu(v_0)$ is expressed in terms of the parameters w_0 and E/p from Eq. (10), and the ratio $[\mathcal{S}_j(w_0)/\mathcal{S}_{j+1}(w_0)]$ is evaluated as a function of w_0. Then for each value of E/p it is possible, in principle, to find a w_0 that yields a value of v_d in Eq. (18) that agrees with the experimental value. Notice in Eq. (18) that the drift velocity depends only implicitly upon the cross section for inelastic collisions through the cutoff energy u_0. The explicit relation between the overshoot and the inelastic cross section will now be found.

IV. INELASTIC CROSS SECTION AND OVERSHOOT

The overshoot (u_0-u_1), determined from the drift-velocity data, can be related to the cross section for inelastic collisions by equating the logarithmic derivatives of the high-energy $(u>u_1)$ distribution function $F_0(u)$ and the low-energy function $f_0(u)$ at u_1. The logarithmic derivative of the low-energy function, which can be obtained directly from Eq. (5), is

$$-\frac{(df_0/du)_{u_1}}{f_0(u_1)} = \frac{f_D(u_1)(dw/du)_{u_1}\int_{u_1}^{u_0}(NQ_m/uf_D)du+NQ_m/u_1}{f_D(u_1)\int_{u_1}^{u_0}(NQ_m/uf_D)du}, \quad (23)$$

where w is defined in Eq. (10). If the overshoot (u_0-u_1) is small,

$$-\frac{(df_0/du)_{u_1}}{f_0(u_1)} \approx \left(\frac{dw}{du}\right)_{u_1}+\frac{1}{(u_0-u_1)}. \quad (24)$$

The logarithmic derivative of the high-energy function $F_0(u)$ is obtained from Eq. (4) by multiplying both sides by dv and integrating from v_1 to infinity. In Eq. (4), $Q_{1m}(v)$ is dropped because it is small compared to $Q_m(v)$, and the primed terms are dropped because only a negligibly small number of inelastic collisions occur at energy greater $2u_1$. When the independent variable is changed to u, the result is

$$-\frac{(dF_0/du)_{u_1}}{F_0(u_1)} = \left(\frac{dw}{du}\right)_{u_1}+\frac{3}{u_1}\left(\frac{1}{eE}\right)^2 NQ_m(u_1) \times\int_{u_1}^\infty \frac{uNQ_1(u)F_0du}{F_0(u_1)}. \quad (25)$$

Hence, by equating Eqs. (24) and (25), the overshoot is

$$(u_0-u_1)^{-1}=[u_1F_0(u_1)]^{-1}\int_{u_1}^\infty Cu(u-u_1)^\gamma F_0(u)du, \quad (26)$$

where $Q_1(u)$ has been replaced by $h(u-u_1)^\gamma$ and $C=3N^2Q_m(u_1)h/e^2E^2$.

The distribution function $F_0(u)$ for the high-energy region is defined by Eq. (4) which can be written in terms of the variable u as

$$\frac{d^2F_0}{du^2}+\left[\frac{d}{du}\left(\ln\frac{u}{NQ_m}\right)+\frac{dw}{du}\right]\frac{dF_0}{du} +\left\{\left[\frac{2}{u}+\frac{d}{du}(\ln NQ_m)\right]\frac{dw}{du}-\frac{3N^2Q_1Q_m}{e^2E^2}\right\}F_0=0. \quad (27)$$

Mathematical difficulties, which are discussed elsewhere,[15] arise in the search for a solution to Eq. (27)

[15] J. C. Bowe, Am. J. Phys. **31**, 905 (1963). This reference presents an elementary but rigorous development of the collision term of the Boltzmann equation. It also discusses the physical significance of the usual approximations that are made in deriving the distribution functions.

that does not ignore elastic collisions and that is also accurate at $u=u_1$.

If the values of E/p are restricted so that $F_0(u)$ rapidly approaches zero for $u>u_1$, then it is not unreasonable to evaluate $Q_m(u)$ at u_1 in Eq. (27). In this case, the solution[15] is

$$F_0(u)=e^{-\alpha u^2}u^{-1/2}W(u)\,, \tag{28}$$

where $4\alpha=A(u_1,E/p)=(6m/M)[NQ_m(u_1)/eE]^2$ and $W(u)$ is defined by

$$(d^2W/du^2)-[K(u)+3N^2Q_1Q_m/e^2E^2]W=0 \tag{29}$$

in which

$$K(u)=(1/4)A^2u^2-A-(1/4u^2)\,. \tag{30}$$

Replacing $(u-u_1)$ in Eq. (26) by Z and substituting for $F_0(u)$ from Eq. (28) leads to

$$(u_0-u_1)^{-1}=C\int_0^\infty Z^\gamma[W(Z)/W(0)]dZ\,, \tag{31}$$

in which u/u_1 was set equal to unity and the terms that multiply $W(u)$ in Eq. (28) were evaluated at u_1. Because of these approximations, Eq. (31) sets a lower limit to the overshoot. Notice that it predicts the correct overshoot of zero and infinity at the limiting cases for which $Q_1(u)$ (i.e.,h) is infinitely large or zero, respectively.

Preliminary computations indicate that the value of C can be as large as about 100 $K(0)$. Therefore, the solution of Eq. (29) obtained by setting $K(u)$ equal to zero is expected to be good also for small values of $Z=(u-u_1)$. This asymptotic solution,[16] in terms of Hankel functions of the first kind, is

$$W(y)=G[(\gamma+2)/2C^{1/2}]^{1/(\gamma+2)}\times y^{1/(\gamma+2)}H_{1/(\gamma+2)}{}^{(1)}(iy)\,, \tag{32}$$

where $y=[2C^{1/2}/(\gamma+2)]Z^{(\gamma+2)/2}$ and G is a constant of integration. The other integration constant is set equal to zero to assure convergence at infinity. At $Z=0$,

$$W(0)=G[(\gamma+2)/2C^{1/2}]^{1/(\gamma+2)}\times\Gamma[1/(\gamma+2)]2^{1/(\gamma+2)}/\pi i^{(\gamma+3)/(\gamma+2)}\,.$$

Equation (31) can now be written

$$(u_0-u_1)^{-1}=[(\gamma+2)^\gamma\, i^{(\gamma+3)}C/2^{(\gamma+1)}]^{1/(\gamma+2)}\times\{\pi/\Gamma[1/(\gamma+2)]\}\int_0^\infty y^{(\gamma+1)/(\gamma+2)}H_{1/(\gamma+2)}{}^{(1)}(iy)dy\,.$$

When the integration[17] is performed, this becomes

$$(u_0-u_1)^{-1}=[(\gamma+2)^\gamma\, C]^{1/(\gamma+2)}\times\{\Gamma[(\gamma+1)/(\gamma+2)]/\Gamma[1/(\gamma+2)]\}\,, \tag{33}$$

which relates the overshoot to the cross section for inelastic collisions.

To see how K affects the overshoot, integrate both sides of Eq. (29) between the limits zero and infinity. Combining this result with Eqs. (31) and (32) yields

$$(u_0-u_1)^{-1}=-W'(0)/W(0)-K\Gamma[2/(\gamma+2)]\div[(\gamma+2)^\gamma C]^{1/(\gamma+2)}\,, \tag{34}$$

where $W'(0)/W(0)$ is the negative of the right-hand side of Eq. (33). Hence, a positive value of the constant K (whose value depends upon E/p) has the effect of increasing the overshoot.

The inaccuracy in the above evaluations of overshoot when $\gamma\neq0$ is due to not knowing the solution to Eq. (29) at $Z=0$. However, the asymptotic solution is expected to be a good representation when $Z\gtrsim[10K/C]^{1/\gamma}$. The accuracy of the overshoot therefore is better for the smaller values of the ratio K/C. Finally, Eq. (33) also predicts that the overshoot is proportional to $(E/p)^{2/(\gamma+2)}$. The results of this paper together with the drift-velocity measurements in the noble gases will be used to evaluate the cross sections for inelastic collisions near threshold.

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[16] E. Jahnke, F. Emde, and F. Lösch, *Tables of Higher Functions* (McGraw-Hill Book Company, Inc., New York, 1960), 6th ed., p. 156.

[17] The integrals may be evaluated by expressing the Hankel functions $H_\nu^{(1)}(iy)$ in terms of modified Bessel functions $K_\nu(y)$ by use of the relations $H_\nu^{(1)}(iy)=-(2i/\pi)e^{-\nu\pi i/2}K_\nu(y)$ and $\int_0^\infty x^{n-1}K_\nu(x)dx=2^{n-2}\Gamma[(n+\nu)/2]\Gamma[(n-\nu)/2]$ for $R(n)>|R(\nu)|$. See, for example, W. Grobner and N. Hofreiter, *Integraltaffel Zweiter Teil Bestimmte Integrale* (Springer-Verlag, Vienna, 1961), Eq. (36) on p. 193 and Eq. (4) on p. 197.